Theorem (Levy) Let $X_t = (X_t^1, \dots, X_t^d)$ be an adapted d-dim process, $X_0 = 0$. Then TFAE.

1) X is d-dimensional Brownian Motion
2) X – continuous local martingale, $\langle X^i, X^j \rangle_t = \delta_{ij} t$ $\forall\ 1 \le i, j \le d$.
3) X – continuous local martingale and $\forall f = (f_1, \dots, f_d) \in L^2(\mathbb{R}_+, \mathbb{R}^d)$, the process
$$\mathcal{E}_t^{if} = \exp\left(i \sum_k \int_0^t f_k \, dX_s^k + \frac{1}{2} \sum_k \int_0^t |f_k^2(s)| ds\right)$$
is a martingale.

Proof. 1) ⟹ 2) by definition.

2) ⟹ 3) $\mathcal{E}_t^{if}$ is the exponential transform of local martingale $M_t = i \sum \int_0^t f_k(s) dX_s^k$.
$$\langle M, M \rangle_t = -\sum \int_0^t |f_k^2(s)| ds$$

So it is a local martingale,
$$|\mathcal{E}_t^{if}| \le \exp\left(\frac{1}{2} \sum_k \|f_k\|_2^2\right) \text{ – bounded!}$$
So it is a martingale.

3) ⟹ 1). Fix $\xi \in \mathbb{R}^d$, $T > 0$ and take $f := 1_{[0,T]} \xi$.

Then
$$\mathcal{E}_t^{if} = \exp\left(i \langle \xi, X_{\min(t,T)} \rangle + \frac{1}{2} |\xi|^2 \min(t,T)\right)$$
inner product

So if $A \in \mathcal{F}_s$ ($s < t < T$) then
$$E(1_A \exp(i \langle \xi, X_t - X_s \rangle)) = P(A) \exp\left(-\frac{|\xi|^2}{2}(t-s)\right)$$
(because $E(\mathcal{E}_t^{if} \mid \mathcal{F}_s) = \mathcal{E}_s^{if}$)

This is true $\forall \xi \in \mathbb{R}^d$, so Fourier transform of $(X_t - X_s)$ is $\exp\left(-\frac{|\xi|^2}{2}(t-s)\right)$ and it is independent of $\mathcal{F}_s$.

So it is Brownian Motion ∎

Corollary. X_t – continuous local martingale (1D!)
$\langle X, X \rangle_t = t$, $X_0 = 0$ ⟹ X_t is a standard Brownian Motion.

Corollary (Complex form). $Z = X + iY$ local martingale,
$\langle Z, Z \rangle_t = 0$, $\langle X, X \rangle_t = t$, $Z_0 = 0$ ⟺ Z – 2D Brownian Motion

Proof $\langle Z, Z \rangle = \langle X, X \rangle - \langle Y, Y \rangle - 2i \langle X, Y \rangle$,

so $\langle Z, Z \rangle_t = 0$, $\langle X, X \rangle_t = t$ ⟺ $\langle X, X \rangle_t = \langle Y, Y \rangle_t = t$, $\langle X, Y \rangle_t = 0$ – can use 2D Levy ∎

Time-changing a process.

Def Time-change is a right-continuous function on $\mathbb{R}_+$ $s \to C_s$, such that each C_s is a stopping time, and $C_s \uparrow$

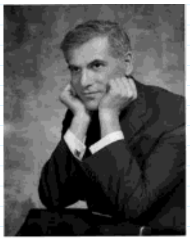
$\hat{X}_s := X_{C_s}$ - time-changed process.

Inverse change: $A_t := \inf\{s : C_s > t\}$, $A_\infty = \lim_{t\to\infty} A_t$

Properties (easy): 1) $\int_{C_0}^{C_t} H_s \, dX_s = \int_0^t 1_{\{C_u < \infty\}} H_{C_u} \, dX_{C_u} = \int_0^{\min(t, A_\infty)} H_{C_u} \, dX_{C_u}$

2) If C_s is a.s finite $\forall s$, X - continuous local martingale then $\langle \hat{X}, \hat{X} \rangle = \widehat{\langle X, X \rangle}$, $\hat{X}$ is continuous local martingale, if $s \to C_s$ - continuous.

3) If H is $(\mathcal{F}_t)$-progressive, $\int_0^t H_s^2 \, d\langle X, X\rangle_s < \infty$ $\forall t$
Then $\int_0^t \hat{H}_s^2 \, d\langle \hat{X}, \hat{X}\rangle_s < \infty$ a.s. $\forall t$ and $\hat{H} \cdot \hat{X} = \widehat{H \cdot X}$



Lester Dubins(1921-2010)

Gideon Schwarz (1933-2007)

Theorem (Dambis, Dubins-Schwarz)
If M - local martingale, $M_0 \equiv 0$ and $\langle M, M\rangle_\infty = \infty$ a.s.
then, for $T_t := \inf\{s : \langle M, M\rangle_s > t\} (= \sup\{s : \langle M, M\rangle_s = t\})$
$B_t := M_{T_t}$ is a Brownian motion,
$M_t = B_{\langle M, M\rangle_t}$

Any local martingale with $\langle M, M\rangle_t \uparrow \infty$ a.s. is a time changed B.M.

Proof The proof uses Flatness lemma:

Proof The proof uses flatness lemma:

Lemma ($\forall t \in [a,b]$ $M_t = M_a$ a.s. on a set $A \subset \Omega$) $\Longleftrightarrow$ $\langle M, M\rangle_b = \langle M, M\rangle_a$ a.s. on A

Proof. ($\Rightarrow$) Just follows from the definition of quadratic variation as a limit.

($\Leftarrow$) Define $N_t := M_t - M_{\min(t,a)}$ - local martingale,

$\langle N, N\rangle_t = \langle M, M\rangle_t - \langle M, M\rangle^a$. $t \geq a$

Fix $\varepsilon > 0$, define $T_\varepsilon = \inf\{t \geq 0, \langle N, N\rangle_t > \varepsilon\}$.

Then $N_0^{T_\varepsilon} = 0$, $\langle N^{T_\varepsilon}, N^{T_\varepsilon}\rangle_\infty = \langle N, N\rangle_{T_\varepsilon} \leq \varepsilon$.

So $N^{T_\varepsilon} \in H^2$, $E((N_t^{T_\varepsilon})^2) \leq \varepsilon$.

On event $A = \{\langle M, M\rangle_b = \langle M, M\rangle_a\}$, $T_\varepsilon > b$.

So $\forall t \in [a,b]$ $E(1_A N_t^2) = E(1_A N^2_{\min(t, T_\varepsilon)}) \leq E((N_t^{T_\varepsilon})^2) \leq \varepsilon$.

Let $\varepsilon \to 0$ to see that

$E(1_A N_t^2) = 0 \Rightarrow N_t = M_t - M_a = 0$ a.s. on A. ∎

$t \to T_t$ - right-continuous, as defined.

$T_t^- = \lim_{\substack{s<t \\ s\to t}} T_s$. Then, by flatness lemma, since $\langle M, M\rangle_{T_t^-} = \langle M, M\rangle_{T_t} = t$, we have

$M_s = M_{T_t}$ for all $T_t^- \leq s \leq T_t$

So $B_t := M_{T_t}$ is a continuous local martingale,

$\langle B, B\rangle_t = \langle M, M\rangle_{T_t} = t$. So, by Levy, B is a BM. ∎

Complex form:

Def. $Z_t = X_t + iY_t$ is conformal local martingale if $\langle Z, Z\rangle_t = 0$ $\forall t$.

Equivalently: Z^2 - also local martingale (since $Z^2 - \langle Z, Z\rangle$ — l.m.)

Equivalently: $\langle X, X\rangle_t = \langle Y, Y\rangle_t$, $\langle X, Y\rangle_t = 0$ $\forall t$.

Observe: If H - progressively measurable, Z - conformal l.m, then $(H\cdot Z)_t = \int_0^t H_s\, dZ_s$ - conformal local martingale

So if F - analytic, then

$$F(Z_t) = F(z_0) + \int_0^t \frac{\partial F}{\partial z}(Z_s)\, dZ_s + \int_0^t \frac{\partial F}{\partial \bar{z}_s}(Z_s)\, d\bar{Z}_s + \frac{1}{4}\int_0^t \Delta F(Z_t)\, d\langle Z, \bar{Z}\rangle_t = F(z_0) + \int_0^t \frac{\partial F}{\partial z}(Z_s)\, dZ_s \text{ - conformal l.m.}$$

Theorem Z - conformal local martingale, $\langle X, X\rangle_\infty = \infty$ as

Then $Z_t = B_{\langle X, X\rangle_t} + z_0$, where B - 2D B.M. started at 0

<u>Proof.</u> By DDS Theorem, $X_t = B_{\langle X,X\rangle_t} + X_0$

Since $\langle Y,Y\rangle_t = \langle X,X\rangle_t$, $Y_t = B^2_{\langle X,X\rangle_t} + Y_0$

Since $\langle X,Y\rangle_t = 0$, we have $\langle B^1, B^2\rangle = 0$.

By Levy, (B^1, B^2) - 2D B.M. ∎

Theorem (Conformal invariance).

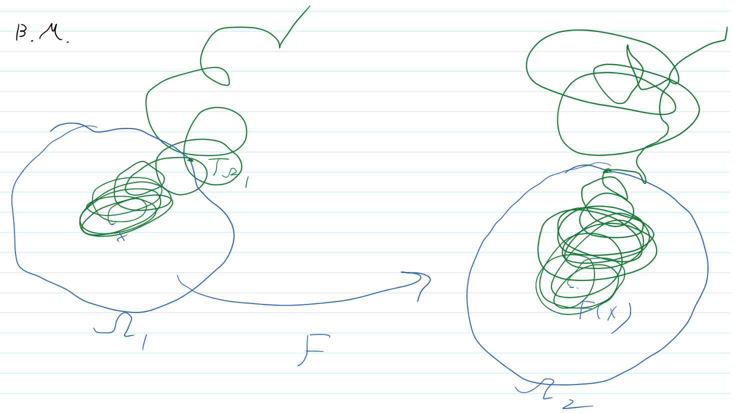
Let $F: \Omega_1 \to \Omega_2$ - conformal, $x \in \Omega_1$, B_t - 2D B.M. started at x; $T_{\Omega_1} = \inf\{t : B_t \notin \Omega_1\}$.

Then $F(B_t)$ is time-changed 2D B.M., i.e. $\exists\ \tilde{B}(t)$ - 2D B.M. started at $F(x)$:

$\forall\ t \le T_{\Omega_1}$, $F(B(t)) = \tilde{B}(s(t))$,

$$s(t) = \int_0^t |F'(B(s))|^2 ds$$

$s(T_{\Omega_1}) = T_{\Omega_2} (= \inf\{t : \tilde{B}_t \notin \Omega_2\})$.



<u>Proof.</u> Let us start with Ω_1, Ω_2 - bounded.

$F(B_t)$ is a conformal martingale (since bounded l.m.)

with $\langle F(B_t), F(B_t)\rangle_t = \int_0^t |F'(B_s)|^2 ds$, by Itô. Strictly increasing, since $|F'| \ne 0$.

As $t \uparrow T_{\Omega_1}$, $F(B_t) \in \Omega_2$, $F(B_t) \to \partial\Omega_2$ - by boundary correspondence

Since Ω_2 is bounded, $F(B_t)$ - bounded, so

a.s. $\exists \lim_{t \to T_{\Omega_1}} F(B_t) \in \partial\Omega_2$, by boundary correspondence

Also, by boundedness, $T := \int_0^{T_{\Omega_1}} |F'(B_s)|^2 ds = \lim_{t \to T_{\Omega_1}} \langle F(B_t), F(B_t)\rangle_t < \infty$ a.s.

<u>Problem</u>: $Z_t = F(B_t^{T_{\Omega_1}})$ does not have $\langle X,X\rangle_\infty = \infty$.

In fact, $\langle X,X\rangle_\infty = T < \infty$ a.s.

Let $(\hat{B}_s)$ - 2D B.M. started at zero, independent of (B_t).

Define $\tilde{B}_s = F(B(\min(t(s), T_{\Omega_1}))) + \hat{B}_s - \hat{B}(\min(s,T))$.

Informally, we attach an independent copy of 2D B.M at $\lim_{t \uparrow T_{\Omega_1}} F(B_t)$.

Then $\tilde{B}_s$ is conformal martingale, as a sum of two independent conformal martingales. $T = \inf\{s : \tilde{B}_s \notin \Omega_2\}$, exit time.

If $\tilde{X}_s = \operatorname{Re} \tilde{B}_s$, $X_s = \operatorname{Re} F(B_{\min(t(s), T_{\Omega_1})})$, then $\hat{X}_s = \operatorname{Re}\hat{B}_s$

$\langle \tilde{X}, \tilde{X}\rangle = \langle X, X\rangle \quad + \langle \hat{X}, \hat{X}\rangle_{.} - \langle \hat{X}, \hat{X}\rangle_{..} \ldots =$

If $\tilde{X}_s = \operatorname{Re} \tilde{B}_s$, $X_s = \operatorname{Re} F(B_{\min(t(s), T_{\Omega_1})})$, then

$\hat{X}_s = \operatorname{Re} \hat{B}_s$

$$\langle \tilde{X}, \tilde{X} \rangle_s = \langle X, X \rangle_{\min(s,T)} + \langle \hat{X}, \hat{X} \rangle_s - \langle \hat{X}, \hat{X} \rangle_{\min(s,T)} =$$
$$\min(s,T) + s - \min(s,T) = s.$$

So $\tilde{B}_s$ is a 2D B.M. started at $F(x)$.

For general Ω_1, Ω_2, consider $\bigcup \Omega_1^n = \Omega_1$, $\bigcup \Omega_2^n = \Omega_2$ - exaustions by bounded domains, $\Omega_2^n = F(\Omega_1^n)$.

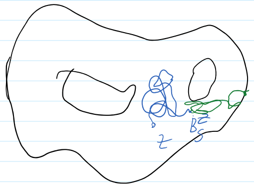
Then Theorem is true up to $T_{\Omega_1^n}$ $\forall n$, and
$T_{\Omega_1^n} \to T_{\Omega_1}$ ∎

Another way to see it:

Theorem (Kakutani). Let $\Omega \subset \mathbb{C}$ be a bounded domain, $f \in C(\partial\Omega)$. For $z \in \Omega$, let (B_t^z) be 2D B.M. started at z. $T_z := \inf\{t : B_t^z \notin \Omega\}$ - exit time.
Then $u(z) = E(f(B_{T_z}^z))$ is the solution of Dirichlet problem with boundary data f.

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \text{ (in some sense)} \end{cases}$$

Proof. Observe: If $S < T_z$ - stopping time, then by OST $E(f(B_{T_z}^z) \mid \mathcal{F}_S) = B_S^z$.
By strong Markov property, B_{S+t}^z - 2D Brownian motion started at B_S^z, $(B_{T_z}^z \mid B_S^z)$ has the same law as $B^{B_S^z}_{T_{B_S^z}}$



So $u(z) = E(f(B_{T_z}^z)) = E(E(f(B_{T_z}^z) \mid \mathcal{F}_S)) =$
$E(u(B_S^z))$.

In particular, if $D(z,r) \subset \Omega$,
$S = \inf\{t : B_t^z \notin D(z,r)\}$, then
by rotation symmetry of Brownian motion,
B_S^z is uniformly distributed on the circle
$C_r = \{w : |w-z| = r\}$.

So $u(z) = E(u(B_S^z)) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta$, so
u is harmonic.

Reminder: Perron solution to Dirichlet problem.

u - solution for Dirichlet problem with boundary values f if

1) $\forall v$ - subharmonic; $\forall \xi \in \partial\Omega$ $\overline{\lim_{z \to \xi}} v(z) \leq f(\xi) \Rightarrow \forall z \in \Omega,\ u(z) \geq v(z)$

2) $\forall v$ - superharmonic: $\forall \xi \in \partial\Omega$ $\underline{\lim_{z \to \xi}} v(z) \geq f(\xi) \Rightarrow \forall z \in \Omega\ u(z) \leq v(z)$.

Now if v is subharmonic, $\overline{\lim_{z \to \xi}} v(z) \leq f(\xi)\ \forall \xi \in \partial\Omega$.

Let $T_n^z := \inf\{t : \operatorname{dist}(B_t^z, \partial\Omega) < \frac{1}{n}\}$ $T_n^z \uparrow T^z$.

Then $v(B^z_{\min(t, T_n)})$ - submartingale (by Itô, if $v \in C^2$, approximate)

$v(z) \leq E(v(B^z_{T_n^z}))$. Take $n \to \infty$, get, by Fatou,

$v(z) \leq E(f(B^z_{T^z})) = u(z)$

Same reasoning shows that for v - superharmonic,

$\underline{\lim_{z \to \xi}} v(z) \geq f(\xi)$, we have $v(z) \geq u(z)$.

So u is Perron solution to Dirichlet problem ∎